Int. J. Solids Structures Vol. 27, No. 13, pp. 1663-1682, 1991
Printed in Great Britain
0020-7683/91 \$3.00 + .00

FRACTURE ANALYSIS OF NONHOMOGENEOUS MATERIALS VIA A MODULI-PERTURBATION APPROACH

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(Received 1 February 1990; in revised form 22 June 1990)

Abstract—This paper presents a first-order moduli-perturbation algorithm for fracture analysis of nonhomogeneous materials. The formulation is based on the Bueckner–Rice weight function theory. In the perturbation procedure a homogeneous body is chosen as the reference so that non-homogeneous quantities are treated as being perturbed from the reference solutions. It is shown that the perturbation formulae can be derived from the potential energy bounds for nonhomogeneous materials, but they generally do not give bounds for estimating stress intensity factors. The perturbation algorithm is applied to calculate the stress intensity factors for several crack problems involving spatially varying material moduli. Comparisons with a few exact solutions indicate that the perturbation results give reasonable predictions over a substantial range of moduli variation. The solution for a cracked body with sinusoidally-varying shear modulus is obtained from perturbation analysis and then used to construct general solutions for arbitrarily-varying modulus via Fourier analysis.

1. INTRODUCTION

Nonhomogeneous materials having elastic moduli that vary with position are either present naturally, or are used intentionally in engineering design to achieve a desired structural performance. Soils, foundations, and geological structures are some examples of these materials occurring naturally while reinforced composites are those being developed intentionally for design purposes. Variations in the effective moduli caused by service damages such as impact and fatigue may also contribute to the nonhomogeneity level of a given material.

Understanding the fracture behavior of nonhomogeneous materials with arbitrarily varying modulus is not only of interest for the technological advance of various types of composite materials, but also for studies of earthquake faulting processes which often involve zones of heterogeneous material properties. However, it is generally very difficult to carry out analytical studies for cracked, nonhomogeneous bodies due to mathematical difficulties. To derive solutions even for plane (2D) or axisymmetric crack problems, it has been necessary to assume special functional forms for the material moduli (e.g. as exponential functions of spatial coordinates; see Dhaliwal and Singh, 1978; Delale and Erdogan, 1983). On the other hand, it is extremely costly to carry out finite element or boundary element analyses, especially when the moduli vary rapidly at short spatial wavelengths.

The lack of an efficient tool in analyzing complex moduli variations justifies the present development of a perturbation approach that allows one to study crack problems in nonhomogeneous materials with more ease. Recent progress in the 3D Bueckner–Rice weight function theory makes it possible to formulate the perturbation algorithm in a general 3D regime. By the unified perturbation procedure to be described in this paper, one may determine the stress intensity factor along a crack front with arbitrary moduli variation, without having to solve the exact boundary value problem. In that procedure a homogeneous body is chosen as the reference so that the nonhomogeneous body is viewed as being perturbed from the reference body via slight perturbations in the material moduli. Two first-order-equivalent perturbation formulae are derived in terms of the 3D weight functions for the reference homogeneous body. It is shown (Appendix) that the two formulae are associated with the potential energy bounds for nonhomogeneous materials, but they generally do not provide bounds for estimating stress intensity factors. Comparison

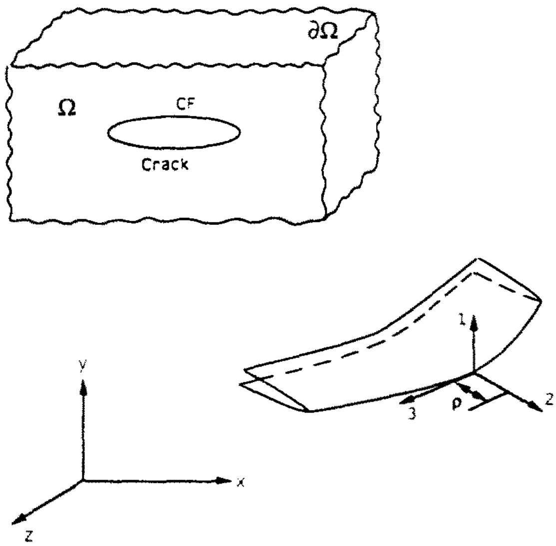

Fig. 1. A 3D planar crack configuration in an elastic body Ω bounded by $\partial\Omega$; the local coordinates 1, 2, 3 along the crack front and the global fixed Cartesian coordinates x, y, z.

with a few exact solutions indicates that the perturbation results are valid over a substantial range of moduli variation. Several nonhomogeneous crack problems of interest are studied for demonstration purposes. Perturbation solutions for cracked bodies with sinusoidally-varying shear modulus are used to construct general solutions via a simple Fourier transform analysis.

2. GENERAL FORMULATION

First consider a body Ω of homogeneous material bounded by a surface $\partial\Omega$. The body carries a displacement field $\mathbf{u}(\mathbf{x})$ generated by some applied forces. Here $\mathbf{x} = (x, y, z)$ is the position vector; bold letters are used for vectors and tensors. The strain $\varepsilon_{ij}(\mathbf{x})$ and stress $\sigma_{ij}(\mathbf{x})$ are given in terms of the displacement field as

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}), \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{1}$$

where commas denote differentiation, C_{ijkl} is the fourth-order elasticity tensor, and Latin subscripts $i, j, k, l, \ldots$ range over the Cartesian directions x, y, z. The given elastic field is in equilibrium with body force $\mathbf{f}(\mathbf{x})$ and boundary traction $\mathbf{t}(\mathbf{x})$, so that the equilibrium equations read

$$\begin{aligned} C_{ijkl}u_{k,li}+f_j &= 0 \quad \text{in} \quad \Omega \\ n_i C_{ijkl}u_{k,l} &= t_j \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2}$$

Here $\mathbf{n}$ is the outer normal along the boundary. The compliance moduli S_{ijkl} may be defined by

$$S_{ijpq}C_{pqkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \quad \text{so that} \quad \varepsilon_{ij} = S_{ijkl}\sigma_{kl}. \tag{3}$$

Assume that the body contains a 3D planar crack. For convenience, a set of local coordinates (Fig. 1) has been adopted at an arc-length location s along the crack front CF. The local axes $\alpha = 1, 2, 3$ are taken to agree with the mode number designations for the local stress intensity factors $K_\alpha(s)$, so that the stress components at a small distance ρ ahead of the crack tip have the asymptotic form

$$\sigma_{1\alpha} \sim K_\alpha(s)/\sqrt{2\pi\rho}. \tag{4}$$

Greek subscripts $\alpha, \beta, \gamma, \ldots$ range over the local directions 1, 2, 3 in contrast to the Latin subscripts $i, j, k, \ldots$ which range over fixed Cartesian coordinates x, y, z. The energy release rate, as energetic force conjugate to crack growth, is given by

$$\mathscr{G}(s) = \Lambda_{\alpha\beta} K_\alpha(s) K_\beta(s). \tag{5}$$

The matrix $\Lambda_{\alpha\beta}$ is symmetric and for isotropic materials:

$$\Lambda_{\alpha\beta} = \mathrm{diag}\left(\frac{1-\nu}{2\mu}, \frac{1-\nu}{2\mu}, \frac{1}{2\mu}\right), \tag{6}$$

where μ, ν are the shear modulus and Poisson's ratio, respectively.

To analyze the crack problems, especially in the 3D regime, it is convenient to employ the weight function theory developed by Bueckner (1970, 1973) and Rice (1972). Recent progress in the development of 3D weight function theory has been reviewed by Rice (1989). The weight functions $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$ are defined as three vector functions of position $\mathbf{x}$ and an observation point s along the crack front: $\mathbf{h}_\alpha = \mathbf{h}_\alpha(\mathbf{x}; s)$. The component $h_{\alpha j}(\mathbf{x}; s)$ corresponds to the mode α stress intensity factor at s due to a unit point force in the j direction at position $\mathbf{x}$. Therefore, the stress intensity factors can be simply constructed as the weighted average of the applied forces with $\mathbf{h}_\alpha$, i.e.

$$K_\alpha(s) = \int_\Omega \mathbf{h}_\alpha(\mathbf{x}; s) \cdot \mathbf{f}(\mathbf{x})\, \mathrm{d}V + \int_{\partial\Omega} \mathbf{h}_\alpha(\mathbf{x}; s) \cdot \mathbf{t}(\mathbf{x})\, \mathrm{d}A. \tag{7}$$

In (7) we have treated the surface forces $\mathbf{t}(\mathbf{x})$ as a Dirac singular layer of body forces along the boundary. The weight function solutions for many 2D crack geometries are listed as point force crack solutions in standard handbooks (e.g. Tada *et al.*, 1985). The 3D solutions for $\mathbf{h}_\alpha$ have been derived for circular cracks and half-plane cracks in an unbounded elastic medium (Bueckner, 1987; Gao, 1989a). Finite element methods have also been developed to compute the 2D and 3D weight functions (e.g. Parks and Kamenetzky, 1979; Sham, 1987) for arbitrary geometry. When the applied forces are specified for a given geometry, one may directly compute the stress intensity factors by the integrals given in (7).

This paper is concerned with cracks in nonhomogeneous materials. We let the elastic moduli C_{ijkl} of the given homogeneous body be perturbed (e.g. via some type of phase transformation) to a spatially variable tensor

$$\tilde{C}_{ijkl}(\mathbf{x}) = C_{ijkl} + \delta C_{ijkl}(\mathbf{x}). \tag{8}$$

The superposed tilde ($\tilde{\ }$) will be used exclusively for quantities pertaining to a nonhomogeneous body. The initial unperturbed homogeneous body acts as a reference system for the perturbed nonhomogeneous body. It will be shown that first-order perturbation solutions can be simply constructed based on the reference solutions $u_i, \sigma_{ij}, \varepsilon_{ij}, K_\alpha$.

In the nonhomogeneous body, the equilibrium equation now reads

$$\tilde{\sigma}_{ij,i} + f_j = 0 \quad \text{with} \quad n_i\tilde{\sigma}_{ij} = t_j \quad \text{on} \quad \partial\Omega, \tag{9}$$

which may be recast into the following form by (8)

$$\begin{aligned} C_{ijkl}\tilde{u}_{k,li} + \{[\delta C_{ijkl}(\mathbf{x})\tilde{u}_{k,l}]_{,i} + f_j\} &= 0 \quad \text{in} \quad \Omega \\ n_i C_{ijkl}\tilde{u}_{k,l} &= \{t_j - n_i\delta C_{ijkl}(\mathbf{x})\tilde{u}_{k,l}\} \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{10}$$

Comparing the above with eqns (2), it is clear that the elasticity problems in a non-homogeneous body are equivalent to those in a homogeneous body subjected to an effective force field within the curly brackets in eqns (10). The force field now depends on the actual elastic field $\tilde{\mathbf{u}}$ weighted by δC_{ijkl}. Since $\tilde{\mathbf{u}}$ is unknown until the given elasticity problem is solved, this observation cannot be used to determine the exact solution of $\tilde{\mathbf{u}}$. However, within the first order accuracy in δC_{ijkl}, one may simply replace $\tilde{\mathbf{u}}$ by $\mathbf{u}$ in the effective force field and write

$$f_i^{\text{eff}} = f_i + (\delta C_{ijkl} u_{k,l})_{,j}, \quad t_i^{\text{eff}} = t_i - n_j \delta C_{ijkl} u_{k,l}, \tag{11}$$

which are fully determined from the homogeneous solutions. Inserting (11) into eqn (7), using the divergence theorem for the perturbation term in f_j^{eff}, which results in cancellation of the boundary integral of the perturbation term in t_i^{eff}, one finds that the stress intensity factor for a nonhomogeneous body can be determined by

$$\tilde{K}_\alpha(s) = K_\alpha(s) - \int_\Omega \delta C_{ijkl}(\mathbf{x}) u_{i,j}(\mathbf{x}) h_{\alpha k,l}(\mathbf{x}\,;s)\,\mathrm{d}V \tag{12}$$

to the first order accuracy in δC_{ijkl}. Therefore, knowledge of the weight function $h_{\alpha k}(\mathbf{x}\,;s)$ for a cracked, homogeneous body permits one to calculate the first-order change in the stress intensity factor due to arbitrary moduli perturbation $\delta C_{ijkl}(\mathbf{x})$.

An alternative first-order formula is obtained by noting that

$$\delta C_{ijkl} = -\tilde{C}_{ijpq}\delta S_{pqmn} C_{mnkl} \simeq -C_{ijpq}\delta S_{pqmn} C_{mnkl}, \tag{13}$$

where $\simeq$ means equal to the first-order accuracy. Hence, within first order accuracy, eqn (12) is equivalent to

$$\tilde{K}_\alpha(s) = K_\alpha(s) + \int_\Omega \delta S_{ijkl}(\mathbf{x}) C_{klpq} \sigma_{ij}(\mathbf{x}) h_{\alpha p,q}(\mathbf{x}\,;s)\,\mathrm{d}V. \tag{14}$$

The first-order perturbation formulae (12), (14) can be also derived from the first-order bounds for the energy change associated with δC_{ijkl}. The derivations are given in the Appendix. Although it is interesting that the present perturbation formulae (12), (14) are somewhat associated with the upper and lower potential energy bounds (Appendix), it is found (Section 4) that they do not provide bounds for estimating the stress intensity factors in a nonhomogeneous body.

A procedure using the moduli perturbation algorithm for fracture analysis in a non-homogeneous material is given as follows. Assume that the elastic moduli for a non-homogeneous body are given as a function $\tilde{C}_{ijkl}(\mathbf{x})$ of the spatial coordinates $\mathbf{x} = x, y, z$. One first chooses a reference body having the same geometry and forces but with a constant moduli value C_{ijkl}. The actual body is then viewed as being perturbed from the reference body by $\delta C_{ijkl}(\mathbf{x}) = \tilde{C}_{ijkl}(\mathbf{x}) - C_{ijkl}$. The stress intensity factors for a nonhomogeneous body are then given by (12) or (14). This perturbation procedure has been used, in a less general sense, by Hutchinson (1987) to analyze a 2D crack with microcracking shielding zones.

The same perturbation procedure applies for estimating the displacements and stresses, since the material inhomogeneity can be represented by the effective force field of (11). Following similar steps leading to the stress intensity formula (12), the displacement field $\tilde{\mathbf{u}}(\mathbf{r})$ can be written as

$$\tilde{u}_m(\mathbf{x}) = u_m(\mathbf{x}) - \int_\Omega \delta C_{ijkl}(\mathbf{x}') u_{i,j}(\mathbf{x}') G_{mk,l}(\mathbf{x}, \mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \tag{15}$$

to first order in δC_{ijkl}, where $G_{mk}(\mathbf{x}, \mathbf{x}')$ denotes the displacement Green's function tensor for the homogeneous body. The stress and strain fields may be computed by differentiation. This procedure was used in a scheme to determine the overall elastic moduli response of composite materials by Willis (e.g. 1983).

In writing eqn (12) we have implicitly assumed that the volume integral involved exists, at least in the sense of the Cauchy principal value if any singularity in the integrand occurs. The singularity at the crack tip presents a potential problem. The volume integral would be well defined if the singularity were weaker than ρ^{-3} everywhere along the crack front, ρ being the distance to the crack tip. For cracks in homogeneous materials, the displacement gradient $u_{i,j} \sim \rho^{-1/2}$ at the crack tip, and the 3D weight functions (Bueckner, 1973) $h_{\alpha i} \sim \rho^{-3/2}$ as the observation point s is approached. The resulting singularity in the integrand is of order ρ^{-3}, which is inadmissible. Hence, eqn (12) is strictly valid only for perturbations that satisfy

$$\delta C_{ijkl}(s) = \lim_{x \to s} \delta C_{ijkl}(x) = 0. \tag{16}$$

A general treatment of the inadmissible singularity in (12) can be taken as follows. For any given $\tilde{C}_{ijkl}(x)$, one has the freedom to choose the reference moduli C_{ijkl} so that condition (16) can always be met, because the reference body is merely a virtual concept in the perturbation scheme. We shall take the reference moduli C_{ijkl} to be equal to $\tilde{C}_{ijkl}(s)$, i.e. the crack-tip value of $\tilde{C}_{ijkl}(x)$, so that

$$\delta C_{ijkl}(x) = \tilde{C}_{ijkl}(x) - C_{ijkl} \quad \text{where } C_{ijkl} = \tilde{C}_{ijkl}(s). \tag{17}$$

From now on, isotropic elastic behavior is assumed where

$$C_{ijkl} = 2\mu\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right], \tag{18}$$

denotes the reference moduli tensor which, by eqn (17), is equal to the moduli value at the crack tip. Following Gao (1989b), we use the following notation:

$$2\mu U^{\alpha}_{ij} = C_{ijkl} h_{\alpha k,l}, \quad 2\mu\,\delta_c U^{\alpha}_{ij} = [\tilde{C}_{ijkl}(x) - C_{ijkl}] h_{\alpha k,l}(x; s), \quad \hat{\sigma}_{ij} = 2\mu\hat{\varepsilon}_{ij}. \tag{19}$$

With these notations, the perturbation formula (12) can be written as

$$\tilde{K}_{\alpha}(s) = K_{\alpha}(s) - \int_{\Omega} \delta_c U^{\alpha}_{ij}(x; s)\,\hat{\sigma}_{ij}(x)\,\mathrm{d}V(x), \tag{20}$$

which is now accurate to first order in $\tilde{C}_{ijkl}(x) - \tilde{C}_{ijkl}(s)$ for any $\tilde{C}_{ijkl}(x)$ that deviates slightly from constancy. The homogeneous crack solutions $\hat{\varepsilon}_{ij}$, $\hat{\sigma}_{ij}$ can be found in standard books for many crack geometries (e.g. Tada *et al.*, 1985; Kanninen and Poplar, 1985). Similarly, the alternative formula (14) can be written as

$$\tilde{K}_{\alpha}(s) = K_{\alpha}(s) + \int_{\Omega} \delta_s \hat{\sigma}_{ij}(x; s)\, U^{\alpha}_{ij}(x; s)\,\mathrm{d}V(x), \tag{21}$$

where

$$\delta_s \hat{\sigma}_{ij} = 2\mu[\tilde{S}_{ijkl}(x) - S_{ijkl}]\sigma_{kl}(x). \tag{22}$$

It can be clearly seen that U^{α}_{ij} are the key quantities in carrying out the perturbation study. For the convenience of further applications, we will present the weight function solutions for $h_{\alpha i}$ and U^{α}_{ij} for a half-plane crack in the next section.

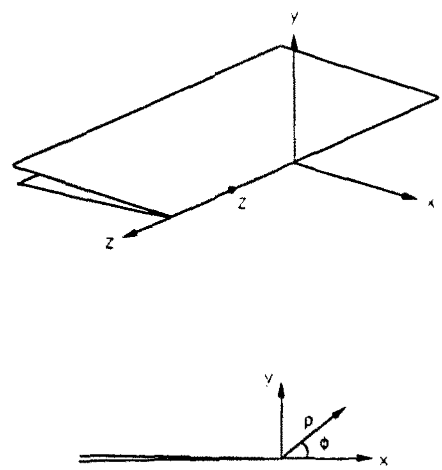

Fig. 2. A half-plane crack in an infinite medium; the Cartesian coordinates x, y, z and the polar coordinates ρ, ϕ at the crack tip.

3. THREE-DIMENSIONAL WEIGHT FUNCTIONS FOR A HALF-PLANE CRACK

Equations (20), (21) have displayed the key role of the weight functions in the moduli-perturbation analysis. Some background will be provided before further applications are made.

Consider a half-plane crack as shown in Fig. 2, with the crack front lying along the z axis. The crack plane occupies the half-plane $y = 0, x < 0$. Bueckner (1987) derived the complete set of weight functions $h_{\alpha j}$ for the half-plane crack for all three crack modes $\alpha = 1, 2, 3$. In his work the weight functions are treated as fundamental fields with higher singularities at crack tips and expressed in terms of a Papkovitch–Neuber potential function

$$G(x, y, z-z') = -\frac{1}{4(1-\nu)\pi^{3/2}}\frac{1}{\zeta}\log\left(\frac{q+\zeta}{q-\zeta}\right), \tag{23}$$

where

$$\zeta = \sqrt{x+i(z-z')};\quad q = \sqrt{2}\,\mathrm{Re}[\sqrt{x+iy}] = \sqrt{2\rho}\cos(\phi/2) \tag{24}$$

and $\mathrm{Re}[F]$ and $\mathrm{Im}[F]$ denote the real and imaginary parts of the complex quantity F. The polar variables ρ, ϕ are defined in the x, y plane as $x+iy = \rho\exp i\phi$. We use complex variable notation during the calculations so that the real parts of the final results are implied for various real quantities such as the stress intensity factors. Let the functions P and Q denote the derivatives of the potential G as

$$P = G_{,y} = -\frac{1}{2(1-\nu)\sqrt{2\pi^3}}\frac{\mathrm{Im}[(x+iy)^{-1/2}]}{\rho-i(z-z')}$$

$$Q = G_{,x}+iG_{,z} = \frac{1}{2(1-\nu)\sqrt{2\pi^3}}\frac{\mathrm{Re}[(x+iy)^{-1/2}]}{\rho-i(z-z')}. \tag{25}$$

The function ζ does not contribute a branch line to the potential function G. Hence G involves only one possible branch line at $x = y = 0$, i.e. the crack front with the branch cut along the crack faces. The real parts of the potential functions G, P, Q are all even functions of $z-z'$, while the real parts of $G_{,z}$, $G_{,zzz}$, etc. are odd functions of $z-z'$. This feature will be used in Section 5 to simplify calculations for a half-crack in a medium with shear modulus varying along the crack front.

The quantities $U^{\alpha}_{mn}(\mathbf{x}; z')$ were presented in terms of the above potential functions by Gao (1989b). For mode I tensile cracks they are

$$U^1_{yy} = P_{,y} - yP_{,yy}, \quad U^1_{yx} = -yP_{,yx},$$
$$U^1_{yz} = -yP_{,yz}, \quad U^1_{xz} = -(1-2\nu)G_{,xz} - yP_{,xz},$$
$$U^1_{xx} = -G_{,xx} + 2\nu G_{,zz} - yP_{,xx}, \quad U^1_{zz} = -G_{,zz} - 2\nu G_{,xx} - yP_{,zz}. \tag{26}$$

Observe that U^1_{xx}, U^1_{yy}, U^1_{yx} are all even functions of $z-z'$, since they only involve quantities like G, P, Q. Also, U^1_{mn} diverges in the order $\rho^{-5/2}$ as the crack front is approached.

For shear modes, i.e. in-plane shear mode 2 and anti-plane shear mode 3, U^{α}_{mn} are written as

$$U^{\gamma}_{yy} = y\psi_{\gamma,yy}, \quad U^{\gamma}_{yx} = -(1-\nu)g_{\gamma,y} + \nu\psi_{\gamma,x} + y\psi_{\gamma,xy},$$
$$U^{\gamma}_{yz} = -(1-\nu)h_{\gamma,y} + \nu\psi_{\gamma,z} + y\psi_{\gamma,zy},$$
$$U^{\gamma}_{xz} = -(1-\nu)(g_{\gamma,z} + h_{\gamma,x}) + y\psi_{\gamma,xz},$$
$$U^{\gamma}_{xx} = -2(1-\nu)g_{\gamma,x} - 2\nu\psi_{\gamma,y} + y\psi_{\gamma,xx},$$
$$U^{\gamma}_{zz} = -2(1-\nu)h_{\gamma,z} - 2\nu\psi_{\gamma,y} + y\psi_{\gamma,zz}, \tag{27}$$

where γ ranges over 2, 3 corresponding to shear modes 2 and 3. The original expressions for the shear potentials g_γ, h_γ, ψ_γ in Bueckner (1987) are presented in a complicated form. They have been simplified by Gao (1989b) to the following:

$$g_2 = -\frac{2P + \nu L_{,x}}{2-\nu}, \quad h_2 = -\frac{\nu L_{,z}}{2-\nu}, \quad \psi_2 = -\frac{2(1-\nu)G_{,x} + \nu Q}{2-\nu},$$
$$g_3 = -\frac{\nu L_{,z}}{2-\nu}, \quad h_3 = -\frac{2(1-\nu)P - \nu L_{,x}}{2-\nu}, \quad \psi_3 = -\frac{2(1-\nu)G_{,z}}{2-\nu}, \tag{28}$$

where $L = 2(yG_{,x} - xG_{,y})$. Observe that the quantities U^2_{xx}, U^2_{yy}, U^2_{yx}, U^3_{zy}, U^3_{zx} are all even functions of $z-z'$. They will play a role in the 3D analysis of a half-plane crack in Section 5.

4. TWO-DIMENSIONAL CRACK ANALYSIS

We first apply the moduli perturbation formulae to 2D crack geometries. Of particular interest here is a 2D semi-infinite crack lying in an infinite medium with its crack tip located in an arbitrarily-shaped nonhomogeneous zone, as shown in Fig. 3a. The material outside the nonhomogeneous zone is assumed to be homogeneous with moduli C^0_{ijkl}. The remote stress field is represented by the stress intensity factor K^0_α. The stress intensity at the crack tip is given by an unknown factor K^{tip}_α, to be determined. One may interpret K^0_α as the apparent stress intensity factor and K^{tip}_α as the real stress intensity factor at the crack tip shielded by the nonhomogeneous zone. Here all the quantities are independent of the variable z so that one may carry out the integration in the z direction in eqns (20), (21) to write

$$K^{\text{tip}}_\alpha = K^0_\alpha - \int_{-\pi}^{\pi}\int_0^{\infty} \delta_c \bar{U}^\alpha_{ij}\,\hat{\sigma}_{ij}\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \tag{29}$$

and

$$K^{\text{tip}}_\alpha = K^0_\alpha + \int_{-\pi}^{\pi}\int_0^{\infty} \delta_s\hat{\sigma}_{ij}\,\bar{U}^\alpha_{ij}\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi, \tag{30}$$

where the polar coordinates ρ, ϕ are set up with origin at the crack tip. For each crack

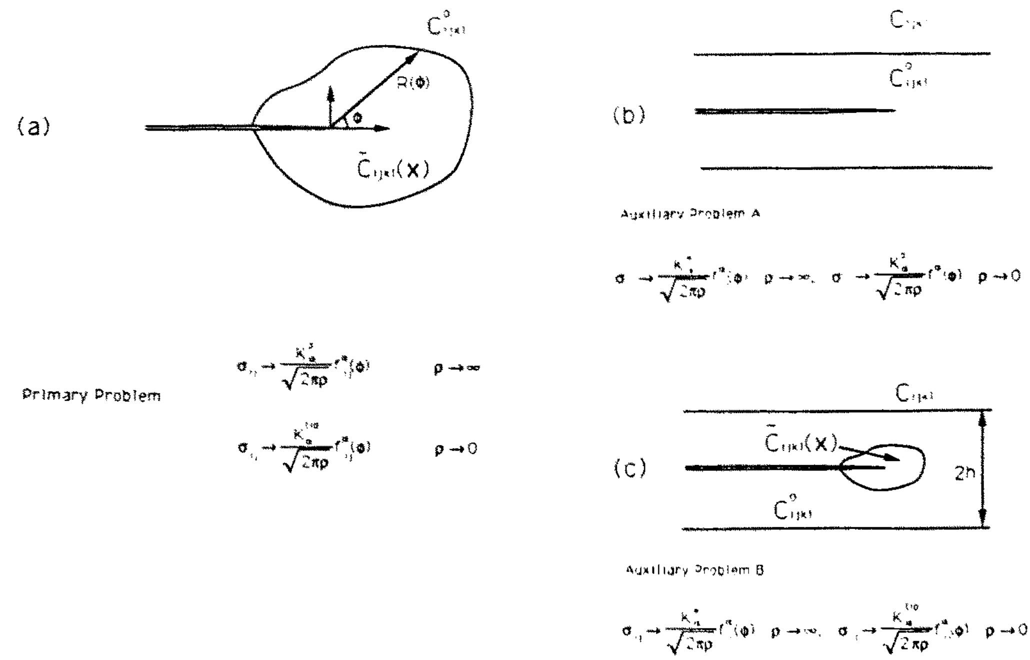

Fig. 3. (a) A semi-infinite crack surrounded by a nonhomogeneous zone. The stress intensity factor at infinity is K^0_α while the crack-tip stress intensity factor is K^{tip}_α. (b) Auxiliary problems A and (c) B for constructing the solution to the primary problem.

mode, the solution for $\hat\sigma_{ij} = 2\mu\varepsilon_{ij}$ can be extracted from the well-known crack fields (e.g. Kanninen and Poplar, 1985).

The 2D quantities $\bar U^\alpha_{kl}$ may be derived by integrating the 3D solutions expressed in (26), (27). In terms of the polar variables ρ, ϕ they are (Gao, 1989b):

$$\bar U^1_{yy} = (C/8)\rho^{-3/2}[7\cos(3\phi/2) - 3\cos(7\phi/2)];$$
$$\bar U^1_{xx} = (C/8)\rho^{-3/2}[\cos(3\phi/2) + 3\cos(7\phi/2)];$$
$$\bar U^1_{yx} = (3C/8)\rho^{-3/2}[\sin(7\phi/2) - \sin(3\phi/2)]; \tag{31}$$

$$\bar U^2_{yy} = \bar U^1_{yx}; \quad \bar U^2_{yx} = \bar U^1_{xx}; \quad \bar U^2_{xx} = -(C/8)\rho^{-3/2}[5\sin(3\phi/2) + 3\sin(7\phi/2)] \tag{32}$$

$$\bar U^3_{yz} = [(1-\nu)C/2]\rho^{-3/2}\cos(3\phi/2); \quad \bar U^3_{xz} = -[(1-\nu)C/2]\rho^{-3/2}\sin(3\phi/2); \tag{33}$$

(the rest of the components $\bar U^\alpha_{kl} = 0$), where $C = 1/[2(1-\nu)(2\pi)^{1/2}]$. The mode I results (31) are in agreement with those used by Hutchinson (1987).

The integrands of eqns (29), (30) behave as ρ^{-1} when $\rho \to \infty$. This leads to logarithmic singularity unless $C^0_{ijkl} = C_{ijkl}$, which shows that the perturbation formulae cannot be directly used for the given primary problem shown in Fig. 3a. One may, however, construct the solution to the primary problem via two auxiliary problems A and B shown in Figs 3b,c. Auxiliary problem A involves an infinite strip with a centered semi-infinite crack. A simple energy argument or application of J-integral gives the exact solution

$$\frac{1-(1-\delta_{\alpha 3})\nu^0}{2\mu^0}(K^0_\alpha)^2 = \frac{1-(1-\delta_{\alpha 3})\nu}{2\mu}(K^*_\alpha)^2 \tag{34}$$

(no summation on α). Auxiliary problem B is immediately solved using the first-order formula (29) or (30). The solution to the primary problem is then constructed as

$$\frac{K^{tip}}{K_\alpha^0} = \left(\frac{K_\alpha^*}{K_\alpha^0}\right)_A \left(\frac{K_\alpha^{tip}}{K_\alpha^*}\right)_B, \tag{35}$$

where the first bracketed term on the right-hand side denotes the solution to auxiliary problem A and the second bracketed term denotes the solution to B. Decomposition (35) may be justified by considering the auxiliary problem B with a sufficiently large strip width h. Letting R be the characteristic length of the nonhomogeneous zone $R(\theta)$, one may identify the following three regions with different stress intensities: (i) region $R \gg \rho \to 0$ with intensity factor K_α^{tip}; (ii) region $h \gg \rho \gg R$ with K_α^0; (iii) region $\rho \gg h$ with K_α^*. Then by an argument similar to that of the small-scale yielding in fracture mechanics, decomposition (35) follows from the fact that

$$\left(\frac{K_\alpha^*}{K_\alpha^0}\right)_B \simeq \left(\frac{K_\alpha^*}{K_\alpha^0}\right)_A, \quad \left(\frac{K_\alpha^{tip}}{K_\alpha^0}\right)_B \simeq \frac{K_\alpha^{tip}}{K_\alpha^0}. \tag{35'}$$

The approximation signs $\simeq$ can be made exact by having the strip width h (a virtual concept) become infinitely large.

For most engineering and geological materials, Poisson's ratio varies only slightly ($\nu = 1/4 \sim 1/3$). If we neglect the variation in Poisson's ratio, the solution to auxiliary problem A is

$$K_\alpha^*/K_\alpha^0 = \sqrt{\mu/\mu^0}. \tag{36}$$

Applying the perturbation formula (29) to auxiliary problem B, and using (31)–(33) and the well-known standard crack fields, it is messy but straightforward to derive the final solutions to the primary problem of Fig. 3a as

$$\frac{K_\alpha^{tip}}{K_\alpha^0} = \sqrt{\frac{\mu}{\mu^0}}\left(1 - \int_{-\pi}^{\pi}\int_0^{R(\phi)} \frac{\tilde{\mu}(\rho,\phi)-\mu}{\mu}\frac{I_\alpha(\phi)}{\rho}\,d\rho\,d\phi - \frac{\mu^0-\mu}{\mu}\int_{-\pi}^{\pi} J_\alpha(\phi)\frac{R'(\phi)\sin\phi + R(\phi)\cos\phi}{R(\phi)}\,d\phi\right), \tag{37}$$

where $I_\alpha(\phi)$ and $J_\alpha(\phi)$ are given by

$$I_1 = \frac{1}{64(1-\nu)\pi}[11\cos\phi + 8\cos 2\phi - 3\cos 3\phi - 16\nu(\cos\phi + \cos 2\phi)]$$

$$I_2 = \frac{1}{64(1-\nu)\pi}[15\cos\phi - 8\cos 2\phi + 9\cos 3\phi - 16\nu(\cos\phi - \cos 2\phi)]$$

$$I_3 = \frac{1}{4\pi}\cos\phi$$

$$J_1 = \frac{1}{32(1-\nu)\pi}[5 + 4\cos\phi - \cos 2\phi - 8\nu(1+\cos\phi)]$$

$$J_2 = \frac{1}{32(1-\nu)\pi}[9 - 4\cos\phi + 3\cos 2\phi - 8\nu(1-\cos\phi)]$$

$$J_3 = \frac{1}{4\pi}. \tag{38}$$

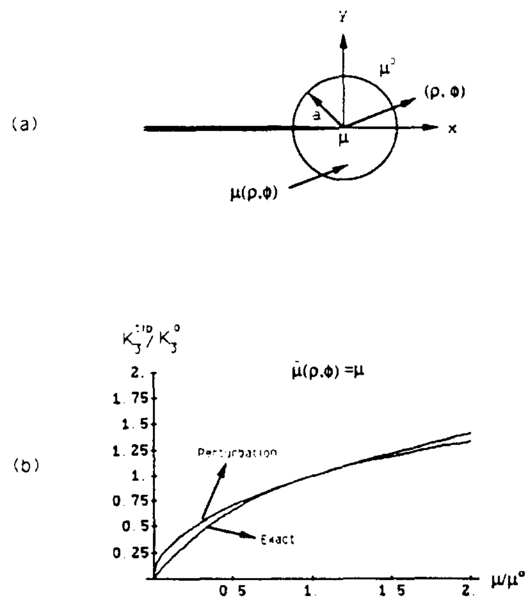

Fig. 4. (a) A semi-infinite crack penetrating a circular inclusion with varying shear modulus $\tilde{\mu}(\rho, \phi)$. (b) Comparison of the perturbation results and the exact solution for a mode 3 crack when $\tilde{\mu}(\rho, \phi)$ is constant.

It may be verified that $I_x(\phi)$ and $J_x(\phi)$ are related by

$$I_x(\phi) = J_x'(\phi)\sin\phi + J_x(\phi)\cos\phi. \tag{39}$$

The above allow one to determine the stress intensity factor $K_x^{\rm tip}$ for an arbitrarily-varying shear modulus $\tilde{\mu} = \tilde{\mu}(\rho, \phi)$ within an arbitrary region $\rho = R(\phi)$ surrounding the crack tip. The mode I part of the solutions in (37)–(38) is consistent with a less general result given by Hutchinson (1987), who treated the special case where the nonhomogeneous zone is symmetric with respect to the crack tip, i.e. where $R(\phi)$ is even in ϕ.

Similarly, application of the alternative perturbation formula (30) results in

$$\frac{K_x^{\rm tip}}{K_x^0} = \sqrt{\frac{\mu}{\mu^0}}\left(1 - \int_{-\pi}^{\pi}\int_0^{R(\phi)} \frac{\tilde{\mu}(\rho,\phi)-\mu}{\tilde{\mu}(\rho,\phi)}\frac{I_x(\phi)}{\rho}\,\mathrm{d}\rho\,\mathrm{d}\phi - \frac{\mu^0-\mu}{\mu^0}\int_{-\pi}^{\pi} J_x(\phi)\frac{R'(\phi)\sin\phi + R(\phi)\cos\phi}{R(\phi)}\,\mathrm{d}\phi\right), \tag{40}$$

which is obviously equivalent to (37) within first order.

Special case: circular nonhomogeneous region at a crack tip

If the nonhomogeneous region is circular, i.e. $R(\phi) = R$ as shown in Fig. 4a, the perturbation formulae simplify to

$$\frac{K_x^{\rm tip}}{K_x^0} = \sqrt{\frac{\mu}{\mu^0}}\left(1 - \int_{-\pi}^{\pi}\int_0^{R} \frac{\tilde{\mu}(\rho,\phi)-\mu}{\mu}\frac{I_x(\phi)}{\rho}\,\mathrm{d}\rho\,\mathrm{d}\phi - \frac{\mu^0-\mu}{\mu}\int_{-\pi}^{\pi} J_x(\phi)\cos\phi\,\mathrm{d}\phi\right) \tag{41}$$

and

$$\frac{K_x^{\rm tip}}{K_x^0} = \sqrt{\frac{\mu}{\mu^0}}\left(1 - \int_{-\pi}^{\pi}\int_0^{R} \frac{\tilde{\mu}(\rho,\phi)-\mu}{\tilde{\mu}(\rho,\phi)}\frac{I_x(\phi)}{\rho}\,\mathrm{d}\rho\,\mathrm{d}\phi - \frac{\mu^0-\mu}{\mu^0}\int_{-\pi}^{\pi} J_x(\phi)\cos\phi\,\mathrm{d}\phi\right) \tag{42}$$

for arbitrarily-varying shear modulus $\tilde{\mu}(\rho, \phi)$ within the circle $\rho = R$.

For the special case of a mode 3 crack penetrating a circular inclusion with constant shear modulus $\tilde{\mu}(\rho, \phi) = \mu$, an exact solution exists which is simply (e.g. Steif, 1987)

$$K_3^{\rm tip}/K_3^0 = 2\mu/(\mu+\mu^0). \tag{43}$$

In this case, both perturbation formulae (41), (42) predict

$$K_3^{\rm tip}/K_3^0 = \sqrt{\mu/\mu^0}. \tag{44}$$

A comparison between (43) and (44) via a plot in Fig. 4b shows that the first-order-accurate perturbation solutions give reasonable results over a substantial range of difference in moduli. Also, note that although the perturbation formulae are associated with potential energy bounds of the nonhomogeneous body (Appendix), they do not provide bounds for estimating the stress intensity factors. In principle, one may choose either one of (37), (40) in application. In practical terms, it is often more convenient to use formula (40) when the crack tip modulus μ is substantially decreased from μ^0 since (37) involves μ in the denominator, which results in larger error than (40).

For mode 1 and mode 2 cracks penetrating a circular inclusion with constant modulus μ, it follows from (42) that

$$\frac{K_1^{\rm tip}}{K_1^0} = \sqrt{\frac{\mu}{\mu^0}}\left(1 - \frac{1-2\nu}{8(1-\nu)}\frac{\mu^0-\mu}{\mu^0}\right),$$

$$\frac{K_2^{\rm tip}}{K_2^0} = \sqrt{\frac{\mu}{\mu^0}}\left(1 + \frac{1-2\nu}{8(1-\nu)}\frac{\mu^0-\mu}{\mu^0}\right). \tag{45}$$

It may easily be verified that these results are consistent, within first-order accuracy, with the asymptotic results of Hutchinson (1987) and Wu (1988). Curiously, the present formulae in the form of (45) are in close agreement with the modified formulae devised by Hutchinson [see eqn (A18) of Hutchinson, 1987] by fitting the exact numerical results. Hutchinson's results have been claimed to agree with the numerical analysis of Steif (1987) on cracks penetrating a circular inclusion. We do not go into more detail here.

It is also interesting to observe that for a circular region with axisymmetric, radially varying moduli $\tilde{\mu} = \tilde{\mu}(\rho)$, the perturbation result $K_\alpha^{\rm tip}$ as calculated from (41) or (42) depends only on the crack tip modulus $\mu = \tilde{\mu}(0)$, because the integral term involving $I_\alpha(\phi)$ vanishes. This suggests that the profile of radially changing moduli within a circular nonhomogeneous region does not affect the crack-tip stress intensity factors. Hence, a circular region with radially-varying moduli can be treated as an inclusion with constant moduli, as observed by Steif (1987) from his numerical result for a mode 3 crack.

Sinusoidally-varying shear modulus

When cracks advance in a material with nonhomogeneous material constants, the crack-tip stress intensity factors change with position in response to the moduli variation. Consider a crack in an infinite strip with height $2h$ (Fig. 5a). Assume that the shear modulus varies sinusoidally in the x direction as

$$\tilde{\mu} = \mu^0 + \mu^A \cos\frac{2\pi x}{\lambda}. \tag{46}$$

When the crack advances along the x axis, the crack tip is located at a position $x = a$, as shown in Fig. 5a. Substituting (46) into the general formula (40), taking $R(\phi) = h/|\sin\phi|$, simplifying the integration with respect to ϕ, and keeping only the first-order terms, one finds that the stress intensity factor at the crack tip behaves as

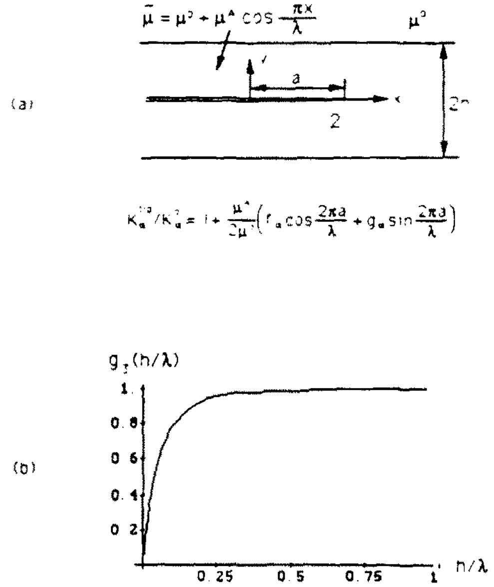

Fig. 5. (a) A crack in an infinite strip with sinusoidally-varying shear modulus. (b) Plot for $g_3(h/\lambda)$ ($f_3 = 1$) that appears in the mode 3 crack solution.

$$K_\alpha^{\rm tip}/K_\alpha^0 = 1+\frac{\mu^A}{2\mu^0}\left[f_\alpha(h/\lambda)\cos\frac{2\pi a}{\lambda}+g_\alpha(h/\lambda)\sin\frac{2\pi a}{\lambda}\right], \tag{47}$$

where

$$f_\alpha(h/\lambda) = 1+4\int_0^\pi \frac{J_\alpha(\phi)}{\cos\phi}\left[1-\cos\left(\frac{2\pi h}{\lambda}\cot\phi\right)\right]\mathrm{d}\phi$$

$$g_\alpha(h/\lambda) = 4\int_0^\pi \frac{J_\alpha(\phi)}{\cos\phi}\sin\left(\frac{2\pi h}{\lambda}\cot\phi\right)\mathrm{d}\phi. \tag{48}$$

The functions $J_\alpha(\phi)$ are those presented in (38). For long-wavelength modulus variations when $h/\lambda \ll 1$, it is easily seen that $f_\alpha \to 1$, $g_\alpha \to 0$ so that

$$K_\alpha^{\rm tip}/K_\alpha^0 = 1+\frac{\mu^A}{2\mu^0}\cos\frac{2\pi a}{\lambda}. \tag{49}$$

The stress intensity factors are in phase with the shear modulus variation. This case is of interest in studying crack behavior in a sandwiched layer with slowly changing moduli.

The other limiting case, $h/\lambda \to \infty$, is of more practical interest, that is, when the strip height approaches infinity or equivalently the modulus wavelength approaches zero. It is found that $f_\alpha(h/\lambda)$, $g_\alpha(h/\lambda)$ quickly approach their limiting values $f_\alpha(\infty)$, $g_\alpha(\infty)$ once h/λ exceeds 0.5, indicating that the strip size does not play a role in the crack-tip stress intensity factors once the wavelength of the shear modulus is smaller than twice the strip height. Only for substantially long wavelengths $h/\lambda \ll 1$ is $K_\alpha^{\rm tip}$ sensitive to the strip size. We have plotted the function $g_3(h/\lambda)$ in Fig. 5b to demonstrate the behavior typical of f_α and g_α. The limiting values $f_\alpha(\infty)$, $g_\alpha(\infty)$ are found to be

$$f_1 = \frac{3-4\nu}{2(1-\nu)},\quad f_2 = \frac{1}{2(1-\nu)},\quad g_1 = g_2 = \frac{3-4\nu}{4(1-\nu)},\quad f_3 = g_3 = 1. \tag{50}$$

The above gives the crack behavior in an unbounded medium with sinusoidally-varying modulus in terms of the homogeneous crack solution K_α^0. Therefore,

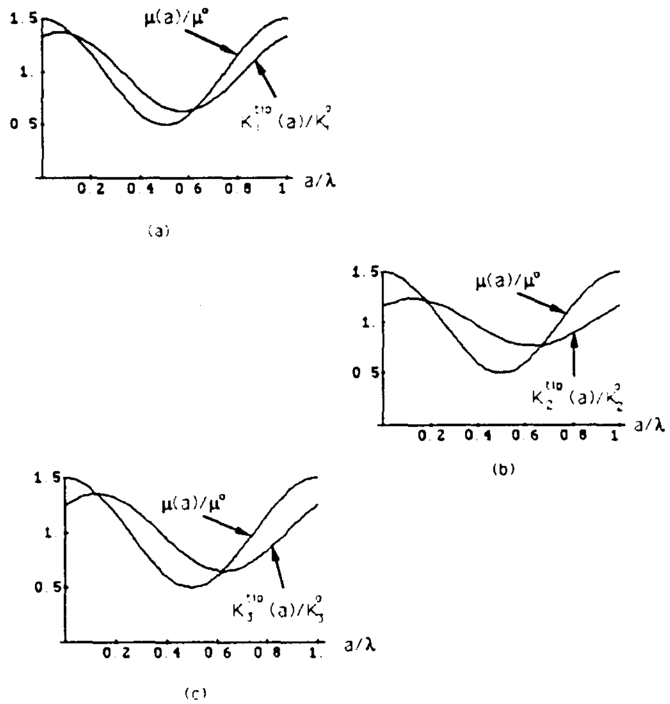

Fig. 6. Solutions to the problem in Fig. 5 when $h/\lambda \to \infty$; crack-tip stress intensity factors versus the local shear modulus profile as a function of the crack tip position a.

$$\frac{K_1^{\text{tip}}}{K_1^0} = 1 + \frac{(3-4\nu)\mu^A}{8(1-\nu)\mu^0}\left(2\cos\frac{2\pi a}{\lambda} + \sin\frac{2\pi a}{\lambda}\right)$$

$$\frac{K_2^{\text{tip}}}{K_2^0} = 1 + \frac{\mu^A}{8(1-\nu)\mu^0}\left(2\cos\frac{2\pi a}{\lambda} + (3-4\nu)\sin\frac{2\pi a}{\lambda}\right)$$

$$\frac{K_3^{\text{tip}}}{K_3^0} = 1 + \frac{\mu^A}{2\mu^0}\left(\cos\frac{2\pi a}{\lambda} + \sin\frac{2\pi a}{\lambda}\right). \tag{51}$$

Here K_γ^0 is interpreted as the apparent stress intensity factor solution when the medium is treated as being homogeneous with shear modulus μ^0. We have plotted in Fig. 6 the variations of K_γ^{tip} with crack tip position along the x axis at $x = a$ against the local modulus variation profile. In Fig. 6, the value of μ^A/μ^0 is taken as 0.5 while Poisson's ratio is taken as 0.25. Note that the stress intensity factor variation is not in phase with the modulus variation, in contrast to the case when $h/\lambda \to 0$ where K_γ^{tip} are completely in phase with $\bar{\mu}$. In the present case $h/\lambda \to \infty$, the mode 1 stress intensity factor suffers a phase shift of 26.6°, while the shear cracks suffer a phase shift of 45°. In the case of mixed mode cracking, the relative amount of each crack mode will be slightly affected by the shear modulus variation. The perturbation amplitude of $K_\gamma^{\text{tip}}/K_\gamma^0$ due to the modulus variation is largest for mode 1 cracks and smallest for mode 2 cracks.

The above result for the sinusoidal modulus variation can be used to construct general solutions for arbitrary modulus variation in the x direction via Fourier transform analysis. Let the shear modulus be written as

$$\bar{\mu}(x) = \mu^0 + \frac{1}{\pi}\int_0^\infty a(\omega)\cos(\omega x) + b(\omega)\sin(\omega x)\,\mathrm{d}\omega, \tag{52}$$

where μ^0 is the reference modulus and $a(\omega)$, $b(\omega)$ are the Fourier cosine and sine transforms defined by

$$a(\omega) = \int_{-\infty}^{\infty} [\tilde{\mu}(x) - \mu^0] \cos(\omega x)\,\mathrm{d}x, \quad b(\omega) = \int_{-\infty}^{\infty} \tilde{\mu}(x) \sin(\omega x)\,\mathrm{d}x. \tag{53}$$

The reference modulus μ^0 is chosen so that $|a(\omega)|/\mu^0$, $|b(\omega)|/\mu^0 \ll 1$; it may, for example, be simply chosen as the mean value of $\tilde{\mu}(x)$.

The following integral result can be easily proved by Cauchy's integral theorem:

$$PV\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{t-a}\,\mathrm{d}t = \pi i\, e^{i\omega a},$$

where "PV" denotes principal value in the Cauchy sense, so that

$$PV\int_{-\infty}^{\infty} \frac{\cos \omega t}{t-a}\,\mathrm{d}t = -\pi \sin \omega a, \quad PV\int_{-\infty}^{\infty} \frac{\sin \omega t}{t-a}\,\mathrm{d}t = \pi \cos \omega a. \tag{54}$$

The perturbation solutions of (51), the integral representation (54) and the Fourier expression (52) allow one to write the mode I stress intensity factor variation due to arbitrary modulus $\tilde{\mu}(x)$ as

$$\frac{K_{\mathrm{I}}^{\mathrm{tip}}}{K_{\mathrm{I}}^{0}} = 1 + \frac{(3-4\nu)}{8\pi(1-\nu)\mu^0}\left[2\int_0^{\infty} a(\omega)\cos \omega a + b(\omega)\sin \omega a\,\mathrm{d}\omega - \frac{1}{\pi}PV\int_{-\infty}^{\infty} \frac{\int_0^{\infty} a(\omega)\cos \omega x + b(\omega)\sin \omega x\,\mathrm{d}\omega}{x-a}\,\mathrm{d}x\right]$$

$$= 1 + \frac{(3-4\nu)}{8(1-\nu)\mu^0}\left[2\tilde{\mu}(a) - 2\mu^0 - \frac{1}{\pi}PV\int_{-\infty}^{\infty} \frac{\tilde{\mu}(t)}{t-a}\,\mathrm{d}t\right]. \tag{55}$$

Hence, it is straightforward to construct the general solutions for all crack modes as

$$\frac{K_{\mathrm{I}}^{\mathrm{tip}}(a)}{K_{\mathrm{I}}^{0}} = 1 + \frac{(3-4\nu)}{8(1-\nu)\mu^0}\left[2\tilde{\mu}(a) - 2\mu^0 - \frac{1}{\pi}PV\int_{-\infty}^{\infty} \frac{\tilde{\mu}(t)}{t-a}\,\mathrm{d}t\right]$$

$$\frac{K_{2}^{\mathrm{tip}}(a)}{K_{2}^{0}} = 1 + \frac{1}{8(1-\nu)\mu^0}\left[2\tilde{\mu}(a) - 2\mu^0 - (3-4\nu)\frac{1}{\pi}PV\int_{-\infty}^{\infty} \frac{\tilde{\mu}(t)}{t-a}\,\mathrm{d}t\right]$$

$$\frac{K_{3}^{\mathrm{tip}}(a)}{K_{3}^{0}} = 1 + \frac{1}{2\mu^0}\left[\tilde{\mu}(a) - \mu^0 - \frac{1}{\pi}PV\int_{-\infty}^{\infty} \frac{\tilde{\mu}(t)}{t-a}\,\mathrm{d}t\right] \tag{56}$$

for any modulus variation $\tilde{\mu}(x)$ in the direction x of crack propagation. The general solutions of (56) may be directly derived from the perturbation formulae (37), (38), taking $R(\phi) = h/|\sin \phi|$. We leave this to the interested readers for verification.

The case is trivial when the shear modulus varies only in the direction perpendicular to the crack plane, i.e. $\tilde{\mu} = \tilde{\mu}(y)$. Application of the perturbation formulae (37), (38) immediately shows that

$$K_{\alpha}^{\mathrm{tip}}/K_{\alpha}^{0} = \sqrt{\mu/\mu_0}.$$

Here μ_0 and K_{α}^{0} represent the apparent values of the shear modulus and stress intensity factors, while μ and $K_{\alpha}^{\mathrm{tip}}$ are the corresponding values at the crack tip. When $\tilde{\mu}(y)$ is an

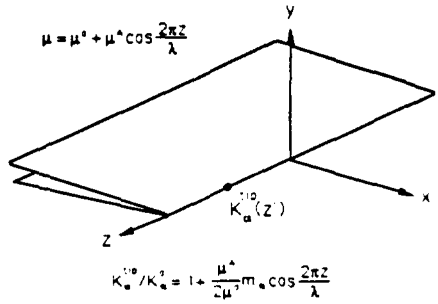

Fig. 7. A half-plane crack in a medium with sinusoidally-varying modulus along the crack front.

even function of y, i.e. when the modulus variation is symmetric about the crack plane, the above perturbation solution becomes exact, as may be verified by a direct energy argument or application of J-integral. However, for general, asymmetric modulus variation, an apparently mode I loading at remote field may induce a mixed mode loading at the crack tip. Such mode coupling, as suggested by our first-order perturbation result, must be of at least second order in the modulus variation. It seems necessary to verify the perturbation result for a generally varying modulus in the y direction via a numerical scheme, but we do not pursue the details here.

5. THREE-DIMENSIONAL APPLICATION

By the unified perturbation procedure given in eqns (20), (21), the 3D crack problems in nonhomogeneous materials can be solved in the same manner as the 2D problems discussed in the last section. For 3D application we consider a half-plane crack front along the z axis (Fig. 7) with the shear modulus varying sinusoidally along the front as

$$\tilde{\mu}(z) = \mu^0 + \mu^\wedge \cos\frac{2\pi z}{\lambda}. \tag{57}$$

The present problem is of interest for the study of earthquake faulting processes that involve heterogeneous zones with varying shear modulus along the fault trace.

To solve the above problem, we first consider the strip-crack problem shown in Fig. 5a with the shear modulus now varying in the z direction. The solution to the original problem in Fig. 7 is obtained by letting the strip height h approach infinity. Note that the stress intensity factors $K_\alpha^{\text{tip}} = K_\alpha^{\text{tip}}(z')$ vary along the crack front with the observation position z'. One may use the perturbation formula (21) to determine $K_\alpha^{\text{tip}}(z')$ in terms of the apparent stress intensity factor K_α^0 at infinity.

The strip problem is solved in the same manner as in deriving (37), (38) for the 2D applications. In the 3D analysis the reference modulus $\mu = \tilde{\mu}(z')$ also varies with the observation point z'. It can be shown that

$$K_\alpha^{\text{tip}}(z') = \sqrt{\tilde{\mu}(z')/\mu^0}\left(K_\alpha^0 - \int_{-\pi}^{\pi}\int_0^{h|\sin\phi|}\int_{-\infty}^{\infty}\frac{\tilde{\mu}(z)-\tilde{\mu}(z')}{\mu}\,\hat{\sigma}_{ij}U_{ij}^\alpha\rho\,\mathrm{d}z\,\mathrm{d}\rho\,\mathrm{d}\phi\right) \tag{58}$$

where $\hat{\sigma}_{ij} = \hat{\sigma}_{ij}(\rho,\phi)$ may be extracted from the well-known crack tip stress fields. Substituting (57) into (58), and observing that the relevant components of U_{ij}^α (e.g. U_{xx}^1, U_{yy}^1, U_{xy}^1) are all even functions of $(z-z')$, one may show that the solution to the strip-crack problem can be written as

$$K_\alpha^{\text{tip}}(z')/K_\alpha^0 = 1+\frac{\mu^\wedge}{2\mu^0}m_\alpha\cos\left(\frac{2\pi z'}{\lambda}\right). \tag{59}$$

where

$$m_\alpha(h/\lambda) = 1 - \frac{4}{K_\alpha^0}\int_0^\pi \int_0^{2\pi h/\lambda \sin\phi} \int_{-\infty}^{\infty} (\cos\tilde{z} - 1)\hat{\sigma}_{ij}(\tilde{\rho},\phi)U_{ij}^\alpha(\tilde{\rho},\phi,\tilde{z})\tilde{\rho}\,\mathrm{d}\tilde{z}\,\mathrm{d}\tilde{\rho}\,\mathrm{d}\phi. \quad (60)$$

In writing eqn (60), we have made the variable transformations $2\pi(z-z')/\lambda = \tilde{z}$, $2\pi\rho/\lambda = \tilde{\rho}$ and used the properties that $\hat{\sigma}_{ij}U_{ij}^\alpha$ is a homogeneous function in $\rho, z-z'$ of degree -3. The functions $m_\alpha = m_\alpha(h/\lambda)$ can be then computed from the 3D weight function solutions given in (26)–(28). The integral in (60) has a finite limit when $h/\lambda \to \infty$, which leads to the solution to the 3D nonhomogeneous crack problem of Fig. 7,

$$m_\alpha = 1 - \frac{4}{K_\alpha^0}\int_0^\pi \int_0^\infty \int_{-\infty}^{\infty} (\cos\tilde{z} - 1)\hat{\sigma}_{ij}(\tilde{\rho},\phi)U_{ij}^\alpha(\tilde{\rho},\phi,\tilde{z})\tilde{\rho}\,\mathrm{d}\tilde{z}\,\mathrm{d}\tilde{\rho}\,\mathrm{d}\phi. \quad (61)$$

The above limiting values of m_α depend only on Poisson's ratio ν, which is in fact obvious from a simple dimensional analysis. Numerical integration may be carried out to compute m_α once ν is given. Our computation for the case $\nu = 0.25$ indicates that

$$m_1 = 1.80, \quad m_2 = 2.11, \quad m_3 = 1.23 \quad (62)$$

for an arbitrarily-varying modulus in the z direction, i.e. $\tilde{\mu} = \tilde{\mu}(z)$. Using a Fourier transform analysis similar to that in Section 4 leads to the following general formula to calculate the stress intensity factors

$$K_\alpha^{\rm tip}(z')/K_\alpha^0 = 1 + \frac{m_\alpha}{2\mu^0}[\tilde{\mu}(z') - \mu^0], \quad (63)$$

where μ^0 is simply the mean value of $\tilde{\mu}$.

It is interesting to note from (63) that the crack-tip stress intensity factors $K_\alpha^{\rm tip}(z')$ only depend on the local modulus at the observation point z'. This suggests that a slight change of moduli such as second-phase inclusions along the crack front will only have very localized effects on the stress intensity factors.

6. DISCUSSION

We have presented a unified moduli-perturbation procedure for determining the first-order-accurate solutions for cracks in a nonhomogeneous medium. Recent developments in the Bueckner–Rice weight function theory in the 3D regime have permitted us to compute the stress intensity factors along a 3D crack front via the perturbation method, without having to solve the complicated boundary value problem. It is shown that although the perturbation formulae are associated with the potential energy bounds of nonhomogeneous materials, they generally do not give bounds for estimating the stress intensity factors. Applications have been made to study semi-infinite crack problems in both 2D and 3D configurations. Comparisons with a few exact solutions available have indicated that the perturbation results are valid over a substantial range of moduli variation.

The present perturbation algorithm can be extended to study many other problems in nonhomogeneous materials. Some of them are listed below.

(a) For reinforced composites, one can use the perturbation algorithm to calculate the interaction between the microcracks in the matrix with second phase fibers or inclusions.

(b) For the analysis of crack growth in composite materials, one can use the perturbation procedure to compute the effects of inclusions or fibers on a major macroscopic crack. For example, it is of interest to study the influence zone of an inclusion within which a crack tip is sufficiently deflected toward the inclusion to cause the so called "crack trapping" process. In the trapping process the crack front advance is blocked, at least temporarily, by a tough inclusion particle. This has been identified as a toughening mechanism for reinforced composites.

(c) Of geological interest, one model for stressing along fault zones involves a slipping shear crack front at the seismogenic zone penetrating upward from depth to cause major earthquakes along tectonic plate boundaries. The geological faults often involve nonhomogeneous zones both along the fault trace and depthwise toward lower portions of the mantle. The perturbation method may open a new channel to study these complex geological problems.

(d) The perturbation method can be also used to assess dislocation interactions with inclusions or arbitrary nonhomogeneous materials. Barnett (1972) has studied a screw dislocation in a nonhomogeneous material with a smoothly varying shear modulus. Generally speaking, very little analytical work is available in this area. The perturbation algorithm may be extended, in the spirit of eqn (15) of the text, to address more complicated issues such as arbitrary dislocation loops interacting with arbitrarily-varying moduli.

It is equally important to note the limitations of the first-order perturbations algorithm:

(a) The perturbation procedure in its present form may not be applicable when the crack front intersects an interface between dissimilar materials, in that an interfacial crack tip generally involves stress singularities different from that of homogeneous cracks, thus leading to singular perturbations. This problem, however, may be overcome by changing the reference solutions from those of a homogeneous crack to those of an interface crack with sufficiently simple moduli variation. As an example, suppose that one wishes to analyze a crack lying along an interface between two drastically dissimilar materials, with the further complication that both materials are weakly nonhomogeneous. Obviously one cannot use a homogeneous body as the reference for perturbation, but it is possible to choose the reference as an interface crack between two materials each having a constant modulus. The philosophy at play here can be summarized as: choose a proper reference body (not necessarily homogeneous) so that the imposed perturbation does not "perturb" the nature of the stress singularity at the crack front location of interest.

(b) For cases where there are drastic variations in material moduli, the first-order-accurate perturbation procedure might cause significant error in estimating the stress intensity factors. In principle, one may construct a multi-step perturbation procedure, in the spirit of the incremental crack front perturbation procedure outlined in Rice (1989), to form a procedure in which the weight function field and the full displacement field are all updated by (12), (15) at each perturbation increment. However, in view of the heavy computations that would be involved in this foreseen procedure, it seems that a more efficient finite element procedure can be devised. The author suspects that the basic concept underneath the present perturbation procedure, namely, representing the material nonhomogeneity by an effective force field [see eqns (10), (11)] can be used to devise a finite element procedure in which the nonhomogeneous cracked body is treated as a homogeneous one with a crack interacting with effective body forces and surface tractions.

Acknowledgements—This work was initiated by the NSF-MRL program through the Center for Material Research at Stanford University and completed with support from the National Science Foundation under Grant No. MSS-9008521. The author gratefully acknowledges a helpful discussion with Professor J. R. Rice during the initial stage of this work. He also thanks two anonymous reviewers for valuable comments on the original manuscript.

REFERENCES

Barnett, D. M. (1972). On the screw dislocation in an inhomogeneous elastic medium: the case of continuously varying elastic moduli. *Int. J. Solids Structures* **8**, 651–660.

Bueckner, H. F. (1970). A novel principle for the computation of stress intensity factors. *Z. Angew. Math. Mech.* **50**, 529–546.

Bueckner, H. F. (1973). Field singularities and related integral representations. In *Mechanics of Fracture 1: Methods of Analysis and Solution of Crack Problems* (Edited by G. C. Sih), pp. 239–314. Noordhoff, Leyden.
Bueckner, H. F. (1987). Weight functions and fundamental fields for the penny-shaped and the half-plane crack in three space. *Int. J. Solids Structures* **23**, 57–93.
Delale, F. and Erdogan, F. (1983). The crack problem for a nonhomogeneous plane. *J. Appl. Mech.* **50**, 609–614.
Dhaliwal, R. S. and Singh, B. M. (1978). On the theory of elasticity of a nonhomogeneous medium. *J. Elasticity* **8**, 211–219.
Eshelby, J. D. (1970). Energy relations and the energy momentum tensor in continuum mechanics. In *Inelastic Behavior of Solids* (Edited by M. F. Kanninen *et al.*), pp. 77–115. McGraw-Hill, Scarborough, CA.
Gao, H. (1989a). Weight functions for external circular cracks. *Int. J. Solids Structures* **25**, 107–127.
Gao, H. (1989b). Applications of 3D weight functions—I. Formulations of crack interactions with transformation strains and dislocations. *J. Mech. Phys. Solids* **37**, 133–153.
Hill, R. (1963). Elastic properties of reinforced solids: some theoretical principles. *J. Mech. Phys. Solids* **11**, 357–372.
Hutchinson, J. W. (1987). Crack tip shielding by micro-cracking in brittle solids. *Acta Metall.* **35**, 1605–1619.
Kanninen, M. F. and Poplar, C. H. (1985). *Advanced Fracture Mechanics*. Oxford University Press, Cambridge.
Parks, D. M. and Kamenetzky, E. A. (1979). Weight functions from virtual crack extensions. *Int. J. Numer. Meth. Engng* **14**, 1693–1706.
Rice, J. R. (1972). Some remarks on elastic crack tip stress fields. *Int. J. Solids Structures* **8**, 751–758.
Rice, J. R. (1985). First order variations in elastic fields due to variation in location of a planar crack front. *J. Appl. Mech.* **52**, 571–579.
Rice, J. R. (1989). Weight function theory for three-dimensional elastic crack analysis. In *Fracture Mechanics: Perspectives and Directions (Twentieth Symposium)*, ASTM STP 1020 (Edited by R. P. Wei and R. P. Gangloff), pp. 29–57. American Society for Testing and Materials, Philadelphia.
Sham, T. L. (1987). A unified finite element method for determining weight functions in two and three dimensions. *Int. J. Solids Structures* **23**, 1357–1372.
Steif, P. S. (1987). A semi-infinite crack partially penetrating a circular inclusion. *J. Appl. Mech.* **54**, 87–92.
Tada, H., Paris, P. C. and Irwin, G. R. (1985). *The Stress Analysis of Cracks Handbook*. Del Research Corporation, Hellertown, PA.
Walpole, L. J. (1970). Strengthening effects in elastic solids. *J. Mech. Phys. Solids* **18**, 343–358.
Willis, J. R. (1983). The overall elastic response of composite materials. *J. Appl. Mech.* **50**, 1202–1209.
Wu, C. H. (1988). A semi-infinite crack penetrating an inclusion. *J. Appl. Mech.* **55**, 736–738.

APPENDIX: PERTURBATION FORMULAE VS ENERGY BOUNDS

For a stressed, homogeneous body Ω, the potential energy and the complementary energy are given by

$$\Phi(\mathbf{u}) = \frac{1}{2}\int_{\Omega}(C_{ijkl}u_{i,j}u_{k,l} - f_i u_i)\,\mathrm{d}V - \int_{\partial\Omega_t} t_i u_i\,\mathrm{d}A$$
$$\Psi(\sigma) = \frac{1}{2}\int_{\Omega} S_{ijkl}\sigma_{ij}\sigma_{kl}\,\mathrm{d}V - \int_{\partial\Omega_u} u_i n_j \sigma_{ij}\,\mathrm{d}A. \tag{A1}$$

Here $\partial\Omega_t$, $\partial\Omega_u$ are the respective portions along the boundary where the tractions or displacements are prescribed. If u_i and σ_{ij} are true solutions to the elasticity problem,

$$\Phi(\mathbf{u}) + \Psi(\sigma) = 0. \tag{A2}$$

The same expressions may be applied to a nonhomogeneous body if all the quantities are superposed with a tilde.

The following energy bound theorem will be used. When a homogeneous body is perturbed to a nonhomogeneous one by changing its moduli by $\delta C_{ijkl}(\mathbf{x})$, the change in the potential energy will be bounded by (e.g. Walpole, 1970)

$$\delta\Phi^{**} \leqslant \delta\Phi \leqslant \delta\Phi^{*}, \tag{A3}$$

where

$$\delta\Phi^{*} = \frac{1}{2}\int_{\Omega}\delta C_{ijkl}(\mathbf{x})u_{i,j}(\mathbf{x})u_{k,l}(\mathbf{x})\,\mathrm{d}V \tag{A4}$$

and

$$\delta\Phi^{**} = -\frac{1}{2}\int_{\Omega}\delta S_{ijkl}(\mathbf{x})\sigma_{ij}(\mathbf{x})\sigma_{kl}(\mathbf{x})\,V. \tag{A5}$$

It is important to note that $\delta\Phi^{*}$ and $\delta\Phi^{**}$ are fully determined by the unperturbed, homogeneous solutions. The above energy bound theorem leads to Hill's (1963) "strengthening theorem" which states that the potential energy always increases when a material is doped with stiffer inclusions, i.e. when δC_{ijkl} $(-\delta S_{ijkl})$ is positive definite. Equally, this also leads to a "weakening theorem" that the energy always decreases when a material is doped with softer inclusion particles, in which case δC_{ijkl} $(-\delta S_{ijkl})$ is negative definite.

The energy bound theorem (A3) can be proved as follows. From the governing equations (1), (2) in the text for the homogeneous solutions $[u_i, \varepsilon_{ij}, \sigma_{ij}]$ and (9) for the nonhomogeneous field $[\tilde{u}_i, \tilde{\varepsilon}_{ij}, \tilde{\sigma}_{ij}]$, one may conclude that $[u_i, \varepsilon_{ij}, \tilde{C}_{ijkl}\varepsilon_{kl}]$ forms a kinematically admissible solution set for the nonhomogeneous body since all the kinematic conditions are met. On the other hand, the set $[\sigma_{ij}, \tilde{S}_{ijkl}\sigma_{kl}]$ is only statically admissible since it satisfies the equilibrium but not the compatibility equations. It follows from the classical minimum energy principles that

$$-\tilde{\Psi}(\sigma) \leqslant -\tilde{\Psi}(\tilde{\sigma}) = \tilde{\Phi}(\tilde{\mathbf{u}}) \leqslant \tilde{\Phi}(\mathbf{u}). \tag{A6}$$

However, it is rather straightforward to show that

$$\tilde{\Phi}(\mathbf{u}) = \Phi(\mathbf{u}) + \delta\Phi^*, \quad \tilde{\Psi}(\sigma) = \Phi(\mathbf{u}) + \delta\Phi^{**}. \tag{A7}$$

Combining eqns (A6) and (A7), one reaches the conclusion that

$$\delta\Phi^{**} \leqslant \delta\Phi = \tilde{\Phi}(\tilde{\mathbf{u}}) - \Phi(\mathbf{u}) \leqslant \delta\Phi^*, \tag{A8}$$

which completes the proof of the energy bound theorem (A3).

Furthermore, it is elementary to show that $\delta\Phi^*$ is equal to $\delta\Phi^{**}$ within first order in δC_{ijkl}. Thus, $\delta\Phi^*$, $\delta\Phi^{**}$ also give the first-order variation in the total potential energy associated with the moduli perturbation $\delta C_{ijkl}(\mathbf{x})$. Without realizing the energy bound theorem (A3), Eshelby (1970) proved the same result via a different approach. According to Eshelby, an analogous result referred to as the Hellman–Feynman theorem also exists in the theory of quantum physics. The present first-order bound theorem has significance in estimating potential energy changes due to phase transformations. We do not pursue the details of that aspect here.

Now assume that the body contains a 3D planar crack. When the crack front CF is perturbed from its original position to a neighboring position by an amount $\delta a(s)$ normal to itself in the crack plane, where $\delta a(s)$ is an arbitrary function of the crack front position s, the potential energy is changed by

$$\delta\Phi^c = -\int_{CF} \mathscr{G}(s)\,\delta a(s)\,\mathrm{d}s, \tag{A9}$$

where $\mathscr{G}(s)$ is the local energy release rate [see eqn (5) of the text]. Therefore, when a homogeneous body is imposed with infinitesimal perturbations $\delta a(s)$ and $\delta C_{ijkl}(\mathbf{x})$, the total energy change is $\delta\Phi^* + \delta\Phi^c$. Writing

$$\delta a(s) = g(s)\,\delta a, \quad \delta C_{ijkl}(\mathbf{x}) = p_{ijkl}(\mathbf{x})\,\delta C \tag{A10}$$

to identify the perturbation magnitudes δa, δC under fixed profile functions $g(s)$ and $p_{ijkl}(\mathbf{x})$, the total energy change is expressed as

$$\delta\Phi = -G\delta a - F^*\delta C \tag{A11}$$

where

$$G = \int_{CF} \Lambda_{\alpha\beta} K_\alpha(s) K_\beta(s) g(s)\,\mathrm{d}s \tag{A12}$$

and

$$F^* = -\frac{1}{2}\int_\Omega p_{ijkl}(\mathbf{x}) u_{i,j}(\mathbf{x}) u_{k,l}(\mathbf{x})\,\mathrm{d}V. \tag{A13}$$

The coefficients G, F^* measure the rates at which the total potential energy varies with the perturbations δa, δC, corresponding to the thermodynamic forces conjugate to the crack advance [with profile $g(s)$] and the moduli change [proportional to $p_{ijkl}(\mathbf{x})$].

The potential energy $\tilde{\Phi}(\tilde{\mathbf{u}})$ depends only on the final elastic state which is a function of variables such as the final crack position and the final moduli function $\tilde{C}_{ijkl}$; it should be independent of the path taken by the perturbation process. Therefore the right-hand side of eqn (A11) is a perfect differential so that the coefficients G and F^* must satisfy the Maxwell reciprocal relation

$$\frac{\partial G}{\partial C} = \frac{\partial F^*}{\partial a} \tag{A14}$$

which by (A12), (A13) leads to

$$2\int_{CF} \Lambda_{\alpha\beta} K_\alpha \frac{\partial K_\beta}{\partial C} g(s)\,\mathrm{d}s = \int_\Omega p_{ijkl}(\mathbf{x}) u_{i,j} \frac{\partial u_{k,l}}{\partial a}\,\mathrm{d}V. \tag{A15}$$

At this stage it is necessary to use an important property of the 3D weight functions $h_{\alpha j}(\mathbf{x}; s)$. Rice (1985) has shown that, when the crack front CF is perturbed by $\delta a(s) = g(s)\delta a$, the displacement $\mathbf{u}$ varies under fixed loading conditions according to

$$\frac{\partial u_k}{\partial a} = 2\int_{CF} \Lambda_{\alpha\beta} h_{\alpha k}(\mathbf{x}; s) K_\beta(s) g(s)\,\mathrm{d}s. \tag{A16}$$

Substituting (A16) into (A15) results in

$$\int_{CF} \Lambda_{\alpha\beta} K_\beta(s) \left\{ \frac{\partial K_\alpha(s)}{\partial C} + \int_\Omega p_{ijkl}(\mathbf{x}) u_{i,j}(\mathbf{x}) h_{\alpha k,l}(\mathbf{x}; s) \, \mathrm{d}V(\mathbf{x}) \right\} g(s) \, \mathrm{d}s = 0. \tag{A17}$$

Equation (A17) must be valid for an arbitrary crack perturbation profile function $g(s)$, which requires that the integrand part within the curly brackets vanishes, i.e.

$$\frac{\partial K_\alpha(s)}{\partial C} = -\int_\Omega p_{ijkl}(\mathbf{x}) u_{i,j}(\mathbf{x}) h_{\alpha k,l}(\mathbf{x}; s) \, \mathrm{d}V(\mathbf{x}). \tag{A18}$$

This equation, when multiplied by δC and replacing $p_{ijkl}(\mathbf{x})\delta C$ by $\delta C_{ijkl}(\mathbf{x})$, leads to the rederivation of eqn (12) of the text. It is then clear that eqn (12) is derived from the upper bound $\delta\Phi^*$ for the energy change due to δC_{ijkl}.

Similarly, if one uses $\delta\Phi^{**}$ instead of $\delta\Phi^*$ in constructing the reciprocal relation (A14) (i.e. F^* is replaced by F^{**}), one can follow similar steps leading to eqn (A18) to rederive eqn (14) of the text. Therefore, the perturbation formulae (12), (14) are in fact associated with the upper and lower first-order bounds for the potential energy change. However, these perturbation formulae do not provide bounds for estimating the stress intensity factors, as shown in Section 4 of the text.